Coupling mechanical systems and fluid transmission lines with bi-dimensional flow and non-conventional constitutive models

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Abstract

The design of hydraulic transmission systems for control and actuation requires accurate knowledge of their dynamic response: some standard techniques are known to obtain a consistent dynamic model of a fluid line, including the contribution of inertia, compressibility and friction. In this study an efficient procedure is developed for simulating the dynamic response of a fluid line coupled with mechanical systems, in both the frequency and time domains. A bi-dimensional approach is adopted for the fluid line, and the laminar flow frequency-dependent friction is modelled using non-integer order differential laws, which may improve the accuracy in comparison with more traditional Newtonian models. The coupling problem with mechanical systems is studied by means of both continuous models of the fluid line (yielding frequency response functions in exact analytical form), and discretized models of the fluid line (to express time response functions in approximate analytical form).

1 Introduction

Fluid transmission lines are employed in both hydraulic control and actuation of mechanical devices in several industrial and automotive applications. For high speed running conditions as well as fast transient operations, accurate models of the dynamic responses of fluid transmission lines are generally required, in order to know in advance critical behaviors and functionality limitations.

Fluid transmission lines have been extensively studied by several authors taking into account attenuation phenomena. These models range from the simplest approach, which assumes a one-dimensional (1-dimensional) uniformly distributed flow [1-2-3], to more complex ones, in which the effects of compressibility and frequency dependent friction are considered as well [4-5].

A relevant problem consists of properly modeling the effects of fluid frequency-dependent friction. In several applications the fluids are mixtures including particles affecting the properties of the resulting liquids, which may be treated as non-Newtonian viscoelastic fluids [6-7]. For this purpose, several analogical models or constitutive equations have been proposed, most of them empirical or semi-empirical. Attempts were made to fit the fluid properties with conventional models, like the integer order Maxwell viscoelastic model, but it was not possible to achieve satisfactory fittings of the experimental data over a wide range of frequencies [8]. Then the development of techniques of analysis including more sophisticated models is of practical interest.

Recently, fractional calculus has encountered much success in the description of viscoelasticity [8-9-10]. The starting point of a non integer derivative model of a non-Newtonian fluid is usually a classical differential equation modified by replacing the integer order time derivatives by fractional differential operators (i.e. non-integer order derivatives, commonly said fractional derivatives).

In this study the transient laminar flow in a circular, constant section fluid line is investigated by means of a bi-dimensional (2-dimensional) approach [11-12-13], modeling the flow frequency-dependent friction using a non-integer order derivative constitutive law, according to the fractional Maxwell model [14-15], of which the Newton model can be considered as a particular case.

The problem of coupling mechanical systems with fluid lines modeled according to the above mentioned non-Newtonian models is studied according to two different approaches: using continuous models, effective

for expressing frequency response functions (Frfs) in exact analytical form, and discretized models by modal decomposition (Galerkin method), effective for expressing time response functions (Trfs) in approximate analytical form.

The developed analytical techniques are discussed by means of numerical examples, showing their accuracy and computational efficiency.

2 Constitutive equations for a fluid line

A non-integer order derivative rheological law of the fluid is considered, according to the linear fractional Maxwell model [7-8-9]. Its constitutive equation is:

$$\tau + \lambda_f \frac{\partial^{\alpha_1}\tau}{\partial t^{\alpha_1}} = \mu_f \frac{\partial^{\alpha_2}\gamma}{\partial t^{\alpha_2}} = \mu_f \frac{\partial^{\alpha_2-1}}{\partial t^{\alpha_2-1}}\left(\frac{\partial\gamma}{\partial t}\right) \tag{1}$$

where τ is the shear stress, γ is the shear strain, its first derivative with respect to time t is the shear rate, λ_f is a fractional relaxation time, μ_f is a fractional viscosity coefficient, α_1 and α_2 are non-integer derivative orders, with $0 \le \alpha_1 \le \alpha_2 \le 1$ [7].

If $\lambda_f = 0$ and $\alpha_2 = 1$, Eq. (1) reduces to the classical Newton model, in which case $\mu_f = \mu$ has the dimension of an absolute viscosity coefficient. If $\lambda_f = 0$ and $\alpha_2 < 1$, Eq. (1) gives the fractional Newton or Scott-Blair model [9]. If $\lambda_f \neq 0$ and $\alpha_1 = \alpha_2 = 1$, Eq. (1) yields the integer order Maxwell model, in which case $\lambda_f = \lambda$ has the dimension of a relaxation time. Laplace transforming Eq. (1) yields:

$$\mathcal{L}(\tau) = \mu(s)\mathcal{L}\left(\frac{\partial\gamma}{\partial t}\right), \quad \mu(s) = \frac{\mu_f}{s^{1-\alpha_2}\left(1+\lambda_f s^{\alpha_1}\right)} \tag{2}$$

in which $\mu(s)$ represents the complex viscosity of the fluid [8]. It is worth noting that the modulus of $\mu(s)$ has the dimension of an absolute viscosity.

3 Equilibrium equations for a fluid line

The 2-dimensional laminar transient flow in a fluid line is modeled according to linearized Navier-Stokes' equations in cylindrical coordinates, as represented in Fig. 1. After introducing some symplifying assumptions and Laplace trasforming as described in [11-12], the equilibrium equations can be written in the following form:

$$\begin{cases} \frac{\partial^2 U_a}{\partial x^2} + \Psi^2 U_a = 0 \\ P_a = -\beta \frac{\partial U_a}{\partial x} \end{cases} \tag{3}$$

where β is the bulk modulus of the fluid, $U_a(x,s)$ and $P_a(x,s)$ are the (cross-section) average values (in the complex domain) of the displacement $u(x,t)$ and pressure $p(x,t)$ respectively, and the complex function $\Psi(s)$ is defined as:

$$\Psi = \frac{s}{c}\sqrt{\frac{\xi_R J_0(\xi_R)}{2J_1(\xi_R) - \xi_R J_0(\xi_R)}}, \quad c = \sqrt{\frac{\beta}{\rho}}, \quad \xi_R = \mathrm{i}R\sqrt{\frac{\rho s}{\mu(s)}} \tag{4}$$

In Eq. (4) ρ and c are the density and the non-viscous speed of sound of the fluid, while $J_0(\cdot)$ and $J_1(\cdot)$ denote Bessel functions of first kind and orders 0 and 1 respectively. The solutions of Eqs. (3) can be expressed in the form:

$$\begin{cases} U_a = A\sin(\Psi x) + B\cos(\Psi x) \\ P_a = \beta\Psi[-A\cos(\Psi x) + B\sin(\Psi x)] \end{cases} \tag{5}$$

where $A(s)$ and $B(s)$ are constants of integration, which can be obtained from two boundary conditions.

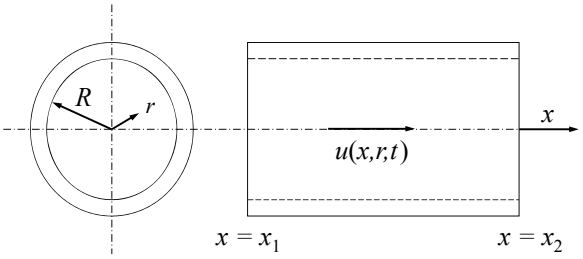

Figure 1: co-ordinate system.

4 Eigenvalue problem for a fluid line

The eigenvalue problem related to the dynamics of a fluid line can be stated by considering the first of Eqs. (3), which can be rewritten in the following form:

$$\frac{1}{U_a}\frac{\partial^2 U_a}{\partial x^2} = -\Psi^2 \tag{6}$$

Since the right-hand side of Eq. (6) does not depend on the variable x, the function $\Psi(s)$ must be stationary. After introducing a set of stationary terms ω_n defined as:

$$\Psi = \frac{\omega_n}{c} \tag{7}$$

the eigenproblem formulation follows recalling Eqs. (3):

$$\begin{cases} \frac{\partial^2 U_a}{\partial x^2} + \frac{\omega_n^2}{c^2} U_a = 0 \\ \omega_n^2 - c^2\Psi^2 = 0 \quad \Leftrightarrow \quad s^2 + 2b(s)s + \omega_n^2 = 0, \;\; b(s) = \frac{J_1(\xi_R)s}{\xi_R J_0(\xi_R) - 2J_1(\xi_R)} \end{cases} \tag{8}$$

The Hermitian function $b(\cdot)$ of the complex variable s depends on the complex viscosity $\mu(s)$, on the density ρ and on the radius R of the fluid line. For the particular case of a Newtonian fluid:

$$\lim_{s\to 0} b(\mathrm{s}) = b(0) = \frac{4\nu}{R^2} \tag{9}$$

which is a well-known result coming from the 1-dimensional transient flow assumption [3].

The stationary terms ω_n introduced in Eq. (7) can be computed from Eqs. (5) by imposing the boundary conditions. In the simplest configurations, and for a fluid line of length $x_2 - x_1 = L$, they are:

$$a)\ \omega_n = n\pi\omega_L, \;\; b)\ \omega_n = \left(n - \frac{1}{2}\right)\pi\omega_L, \;\; \omega_L = \frac{c}{L}, \;\; n \in \mathbb{N} \tag{10}$$

where *a*) holds for a closed-closed or open-open fluid line, *b*) for a closed-open fluid line.

5 Discretized eigenvalue problem for a fluid line

A discretization of the fluid line can be obtained by approximating the flow average displacement U_a by means of a linear combination of eigenfunctions computed with standard boundary conditions as in Eqs. (5) and (10):

$$\begin{cases} U_a(x) \cong \sum_{n=1}^{N} [A_n \sin(n\varphi) + B_n \cos(n\varphi)] = \mathbf{n}^{\mathrm{T}}\mathbf{u} \\ \varphi = \frac{\pi x}{2L} \\ \mathbf{n}^{\mathrm{T}}(x) = [\sin(\varphi), \sin(2\varphi), \dots, \sin(N\varphi), \cos(\varphi), \cos(2\varphi), \dots, \cos(N\varphi)] \\ \mathbf{u}^{\mathrm{T}} = [A_1, A_2, \dots, A_N, B_1, B_2, \dots, B_N] \end{cases} \tag{11}$$

The application of the Galerkin method to the first of Eqs. (8) on the basis of Eqs. (11) yields the following algebraic eigenproblem:

$$\begin{cases} \left[\frac{\widetilde{\omega}^2}{\omega_L^2}\mathbf{M}_L + \mathbf{K}\right]\mathbf{u} = \mathbf{0} \\ \mathbf{M}_L = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathbf{n}\mathbf{n}^{\mathrm{T}} d\varphi\,, \;\; \mathbf{K} = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \mathbf{n}\frac{d^2\mathbf{n}^{\mathrm{T}}}{d\varphi^2} d\varphi \\ s^2 + 2b(s)s + \widetilde{\omega}_n^2 = 0 \end{cases} \tag{12}$$

where $\mathbf{M}_L$ and $\mathbf{K}$ are $2N\times 2N$ dimensionless mass and stiffness matrices respectively. The stiffness matrix $\mathbf{K}$ can be rewritten integrating by parts:

$$\mathbf{K} = \frac{\pi}{2}\left[\left|\mathbf{n}\frac{d\mathbf{n}^{\mathrm{T}}}{d\varphi}\right|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{d\mathbf{n}}{d\varphi}\frac{d\mathbf{n}^{\mathrm{T}}}{d\varphi} d\varphi\right] = \mathbf{K}_{BC} - \mathbf{K}_L \tag{13}$$

which makes it possible to take into account explicitly the boundary conditions (in matrix $\mathbf{K}_{BC}$) and gives a symmetric stiffness matrix of the discretized fluid line ($\mathbf{K}_L$). For the simplest standard boundary conditions (open-open, closed-open, closed-closed ends) $\mathbf{K}_{BC} = \mathbf{0}$. The matrices $\mathbf{M}_L$ and $\mathbf{K}_L$ can be analytically defined as:

$$\begin{cases} \mathrm{M}_{Lnm}|_{1\le m\le N}^{1\le n\le N} = \frac{2}{\pi}\mathrm{SS}_{nm} \quad , \quad \mathrm{K}_{Lnm}|_{1\le m\le N}^{1\le n\le N} = \frac{\pi}{2}nm\mathrm{CC}_{nm} \\ \mathrm{M}_{Lnm}|_{N+1\le m\le 2N}^{1\le n\le N} = \frac{2}{\pi}\mathrm{SC}_{nm}, \quad \mathrm{K}_{Lnm}|_{1\le m\le N}^{1\le n\le N} = -\frac{\pi}{2}nm\mathrm{CS}_{nm} \\ \mathrm{M}_{Lnm}|_{1\le m\le N}^{N+1\le n\le 2N} = \frac{2}{\pi}\mathrm{CS}_{nm}\,, \quad \mathrm{K}_{Lnm}|_{1\le m\le N}^{1\le n\le N} = -\frac{\pi}{2}nm\mathrm{SC}_{nm} \\ \mathrm{M}_{Lnm}|_{N+1\le m\le 2N}^{N+1\le n\le 2N} = \frac{2}{\pi}\mathrm{CC}_{nm}, \quad \mathrm{K}_{Lnm}|_{N+1\le m\le 2N}^{N+1\le n\le 2N} = \frac{\pi}{2}nm\mathrm{SS}_{nm} \end{cases} \tag{14}$$

with:

$$\begin{cases} \mathrm{SS}_{nm} = \frac{1}{2}\left\{\frac{\sin\left[\frac{\pi}{2}(n-m)\right]}{n-m} - \frac{\sin\left[\frac{\pi}{2}(m+n)\right]}{m+n}\right\}, \quad \mathrm{SS}_{nn} = \frac{\pi}{4} \\ \mathrm{CC}_{nm} = \frac{1}{2}\left\{\frac{\sin\left[\frac{\pi}{2}(n-m)\right]}{n-m} + \frac{\sin\left[\frac{\pi}{2}(m+n)\right]}{m+n}\right\}, \quad \mathrm{CC}_{nn} = \frac{\pi}{4} \\ \mathrm{SC}_{nm} = \frac{1}{2}\left\{\frac{1+\cos\left[\frac{\pi}{2}(m+n)\right]}{m+n} + \frac{1-\cos\left[\frac{\pi}{2}(n-m)\right]}{n-m}\right\}, \quad \mathrm{SC}_{nn} = \frac{1-(-1)^n}{4n} \\ \mathrm{CS}_{nm} = \frac{1}{2}\left\{\frac{1+\cos\left[\frac{\pi}{2}(m+n)\right]}{m+n} - \frac{1-\cos\left[\frac{\pi}{2}(n-m)\right]}{n-m}\right\}, \quad \mathrm{CS}_{nn} = \frac{1-(-1)^n}{4n} \end{cases} \tag{15}$$

The algebraic eigenproblem in Eqs. (12) can be written in a different form, introducing a damping matrix $\mathbf{C}_L$:

$$\left[\frac{s^2}{\omega_L^2}\mathbf{M}_L + 2\frac{s}{\omega_L}\frac{b(s)}{b(0)}\mathbf{C}_L + (\mathbf{K}_L - \mathbf{K}_{BC})\right]\mathbf{u} = \mathbf{0}, \;\; \mathbf{C}_L = \frac{b(0)}{\omega_L}\mathbf{M}_L \tag{16}$$

6 Coupling with mechanical systems: solution in the frequency domain

A mechanical discrete linear system with H dofs is considered, with mass, general viscous damping and stiffness $H\times H$ matrices $\widehat{\mathbf{M}}$, $\widehat{\mathbf{C}}$, $\widehat{\mathbf{K}}$ respectively. The transfer function between its m-th and n-th dofs can be conventionally expressed as a linear combination by means of its eigenvalues s_h and normalized eigenvectors $\mathbf{v}^{(h)}$:

$$H_{mn}(s) = \sum_{h=1}^{2H} \frac{v_m^{(h)} v_n^{(h)}}{s - s_h} \tag{17}$$

Let $\hat{\mathbf{x}}$ be the related displacement vector, and assume that its n-th dof $\hat{x}_n$ is coupled with a fluid line. If the external force vector is null except for its n-th element F_n, due to coupling with the fluid line, then the n-th dof equilibrium equation can be written in the Laplace domain in the form:

$$\frac{\hat{X}_n}{F_n} = H_{nn}(s) \tag{18}$$

where $\hat{X}_n$ denotes the Laplace transform of $\hat{x}_n$. Recalling the second of Eqs. (3), the continuity with the fluid line in $x = L$ can be imposed by setting:

$$\begin{cases} \hat{X}_n = U_a(L) \\ F_n = -k_L L \left.\frac{\partial U_a}{\partial x}\right|_L, \quad k_L = \frac{\beta \pi R^2}{L} \end{cases} \tag{19}$$

A second bounday condition for the fluid line must be given in addition to Eq. (19): their combination yields the characteristic equation of the coupled system. As an example, if in $x = 0$ the fluid line is closed, then Eqs. (19) and (5) give:

$$\frac{\hat{X}_n}{F_n} = -\frac{\tan(\Psi L)}{k_L \Psi L} \tag{20}$$

which substituted in Eq. (18) yields the characteristic equation.

Different boundary conditions in $x = 0$ can be considered as well: coupling with a different dof of the same mechanical system (say the q-th dof), or coupling with another separate linear mechanical system (say in its q-th dof). In any case a system of two equations can be written:

$$\begin{Bmatrix} \hat{X}_n \\ \hat{X}_q \end{Bmatrix} = \mathbf{H} \begin{Bmatrix} F_n \\ F_q \end{Bmatrix}, \quad \mathbf{H} = \begin{bmatrix} H_{nn}(s) & H_{nq}(s) \\ H_{qn}(s) & H_{qq}(s) \end{bmatrix} \tag{21}$$

where the matrix $\mathbf{H}$ is diagonal in the case of coupling with two separate linear mechanical systems. Equations (21) can be rewritten introducing the continuity between displacements at the ends of the fluid line:

$$\begin{Bmatrix} U_a(L) \\ U_a(0) \end{Bmatrix} = k_L L \mathbf{H} \begin{Bmatrix} -\left.\frac{\partial U_a}{\partial x}\right|_L \\ \left.\frac{\partial U_a}{\partial x}\right|_0 \end{Bmatrix} \tag{22}$$

from which, recalling the first of Eqs. (5), the characteristic equation follows immediately:

$$\det\left[\begin{bmatrix} \sin(\Psi L) & \cos(\Psi L) \\ 0 & 1 \end{bmatrix} + k_L \Psi L \mathbf{H} \begin{bmatrix} \cos(\Psi L) & -\sin(\Psi L) \\ -1 & 0 \end{bmatrix}\right] = 0 \tag{23}$$

The eigenvalues of the resulting coupled system can be computed by means of a zero-finding routine applied to Eq. (23).

If the mechanical system $\hat{\mathbf{M}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{K}}$ is coupled with a fluid line only in its n-th dof, and if an impulse force of amplitude F_m acting on its m-th dof is introduced, then the associated transfer function between the p-th and m-th dofs of the mechanical system can be written as:

$$\begin{cases} \frac{X_p}{F_m} = H_{pm}(s) + H_L(s) \\ H_L(s) = H_{pn}(s) H_{nm}(s) \left[\frac{\hat{X}_n}{F_n} - H_{nn}(s)\right]^{-1} \end{cases} \tag{24}$$

where $H_L(s)$ represents the contribution to the transfer function of coupling with a fluid line.

Replacing s with $i\omega$ in Eqs. (24) gives the corresponding Frf (in analytical form), whose inverse Fourier transform yields the Irf (in numerical form):

$$s \to \mathrm{i}\omega \quad \Rightarrow \quad H(s) \to H(\omega), \quad h(t) = \mathcal{F}^{-1}[H(\omega)] \tag{25}$$

For the analytical expression of the transfer functions as in Eq. (24), the eigenvalues of the uncoupled mechanical system are needed, and not those of the coupled resulting system, allowing an efficient computation of the Frfs. On the other hand, the computation of the eigenvalues of the coupled resulting system, and of the inverse Fourier transform of the Frfs, require numerical procedures which in general may not be so straightforward. Therefore in the following the advantages-disadvantages of considering a discretized model for the fluid line and the possibility of expressing Irfs in analytical approximate form will be studied and discussed.

7 Coupling with mechanical systems: discretized model

A mechanical discrete linear system with H dofs is considered, as in section 6, whose n-th dof is coupled to a fluid line. Let $\breve{\mathbf{M}}$, $\breve{\mathbf{C}}$, $\breve{\mathbf{K}}$ be three reduced order $(H-1)\times(H-1)$ matrices, obtained from $\widehat{\mathbf{M}}$, $\widehat{\mathbf{C}}$, $\widehat{\mathbf{K}}$ by deleting both their n-th rows and their n-th columns, and $\breve{\mathbf{x}}$ a reduced order displacement vector with $H-1$ elements, obtained from $\hat{\mathbf{x}}$ by Laplace transforming and deleting its n-th component $\hat{X}_n$. In addition, let $\breve{\mathbf{a}}_{c-n}$ denote the n-th column of a matrix $\widehat{\mathbf{A}}$, reduced to $H-1$ components by deleting its n-th component (i.e. deleting $\widehat{\mathrm{A}}_{nn}$), and let $\breve{\mathbf{a}}_{r-n}$ denote the n-th row of a matrix $\widehat{\mathbf{A}}$, reduced to $H-1$ components by deleting $\widehat{\mathrm{A}}_{nn}$. Adopting this notation, the resulting system of equilibrium equations and boundary conditions for the coupled system can be written in the following form:

$$\begin{cases} \left(\mathrm{s}^2\breve{\mathbf{M}} + s\breve{\mathbf{C}} + \breve{\mathbf{K}}\right)\breve{\mathbf{x}} + \left(\mathrm{s}^2\breve{\mathbf{m}}_{c-n} + s\breve{\mathbf{c}}_{c-n} + \breve{\mathbf{k}}_{c-n}\right)\hat{X}_n = \mathbf{0} \\ \left(\mathrm{s}^2\breve{\mathbf{m}}_{r-n} + s\breve{\mathbf{c}}_{r-n} + \breve{\mathbf{k}}_{r-n}\right)\breve{\mathbf{x}} + \left(\mathrm{s}^2\widehat{\mathrm{M}}_{nn} + s\widehat{\mathrm{C}}_{nn} + \widehat{\mathrm{K}}_{nn}\right)\hat{X}_n = F_n = -k_L L\left.\frac{d\mathbf{n}^{\mathrm{T}}}{dx}\right|_L \mathbf{u} \\ \hat{X}_n = \mathbf{n}_L^{\mathrm{T}}\mathbf{u} \\ \left[s^2 m_L\mathbf{M}_L + sc_L\frac{b(s)}{b(0)}\mathbf{C}_L + k_L(\mathbf{K}_L - \mathbf{K}_{BC})\right]\mathbf{u} = \mathbf{0}, \;\; m_L = \rho\pi R^2 L, \;\; c_L = 2\omega_L m_L, \;\; \omega_L = \sqrt{\frac{k_L}{m_L}} \end{cases} \tag{26}$$

where the last equation is Eq. (16) rewritten in a non-dimensionless form, as a function of m_L, c_L, and k_L.

A further boundary condition at the opposite side of the fluid line must be added to Eqs. (26): for simplicity it is assumed a closed end, but as said in section 6 the results could be easily extended to the general case. For a fluid line with a closed end in $x = 0$ and a coupled end in $x = L$, the shape functions are given by setting $B_n = 0$ in the first of Eqs. (11), yielding a set of admissible functions.

The procedure to obtain the $H + N - 1$ equilibrium equations of the coupled system (where N is the selected number of dofs for the discretized fluid line) consists of 5 steps: 1) introducing the third of Eqs. (26) in the second one; 2) multiplying the latter equation to the left by $\mathbf{n}_L$; 3) introducing in the same equation the changement of variables:

$$\frac{d\mathbf{n}^{\mathrm{T}}}{dx} = \frac{\pi}{2L}\frac{d\mathbf{n}^{\mathrm{T}}}{d\varphi} \tag{27}$$

4) recalling the definition of matrix $\mathbf{K}_{BC}$ given in Eq. (13), depending on the boundary conditions of the fluid line, which yields:

$$-k_L\mathbf{K}_{BC} = \left[\mathrm{s}^2\widehat{\mathrm{M}}_{nn} + s\widehat{\mathrm{C}}_{nn} + \widehat{\mathrm{K}}_{nn}\right]\mathbf{n}_L\mathbf{n}_L^{\mathrm{T}} + \mathbf{n}_L\left[\mathrm{s}^2\hat{\mathbf{m}}_{r-n} + s\hat{\mathbf{c}}_{r-n} + \hat{\mathbf{k}}_{r-n}\right]\breve{\mathbf{x}}\mathbf{u}^{\mathrm{T}}\|\mathbf{u}\|^{-2} \tag{28}$$

5) substituting the expression of matrix $\mathbf{K}_{BC}$ given by Eq. (28) in the last of Eqs. (26). After introducing the definition of the following mass, damping and stiffness $N\times N$ matrices:

$$\mathbf{M} = \widehat{\mathrm{M}}_{nn}\mathbf{n}_L\mathbf{n}_L^{\mathrm{T}} + m_L\mathbf{M}_L, \;\; \mathbf{C} = \widehat{\mathrm{C}}_{nn}\mathbf{n}_L\mathbf{n}_L^{\mathrm{T}} + c_L\frac{b(s)}{b(0)}\mathbf{C}_L, \;\; \mathbf{K} = \widehat{\mathrm{K}}_{nn}\mathbf{n}_L\mathbf{n}_L^{\mathrm{T}} + k_L\mathbf{K}_L \tag{29}$$

the resulting $(H+N-1)\times(H+N-1)$ matrices for the coupled system take the form:

$$\mathbf{M}_S = \begin{bmatrix} \breve{\mathbf{M}} & \breve{\mathbf{m}}_{c-n}\mathbf{n}_L^{\mathrm{T}} \\ \mathbf{n}_L\breve{\mathbf{m}}_{r-n} & \mathbf{M} \end{bmatrix}, \; \mathbf{C}_S = \begin{bmatrix} \breve{\mathbf{C}} & \breve{\mathbf{c}}_{c-n}\mathbf{n}_L^{\mathrm{T}} \\ \mathbf{n}_L\breve{\mathbf{c}}_{r-n} & \mathbf{C} \end{bmatrix}, \; \mathbf{K}_S = \begin{bmatrix} \breve{\mathbf{K}} & \breve{\mathbf{k}}_{c-n}\mathbf{n}_L^{\mathrm{T}} \\ \mathbf{n}_L\breve{\mathbf{k}}_{r-n} & \mathbf{K} \end{bmatrix} \tag{30}$$

leading to the $H + N - 1$ equilibrium equations of the coupled system:

$$[s^2\mathbf{M}_S + s\mathbf{C}_S + \mathbf{K}_S]\mathbf{y} = \mathbf{0}, \quad \mathbf{y} = \begin{bmatrix}\check{\mathbf{x}}\\ \mathbf{u}\end{bmatrix} \tag{31}$$

It can be outlined that setting $\widehat{\mathrm{M}}_{nn} = 0$ in the equilibrium equations, yields a viscoelastic link at the interface between fluid and solid systems. In the discretized model the only effect would be having $\mathbf{M} = m_L\mathbf{M}_L$ in Eqs. (29). The matrices $\mathbf{M}_S$, $\mathbf{C}_S$ and $\mathbf{K}_S$ are symmetric, $\mathbf{M}_S$ definite positive, $\mathbf{C}_S$ and $\mathbf{K}_S$ definite positive or semi-definite positive. $\mathbf{C}_S$ does not depend on the eigenvalue s only in the case of the 1-dimensional Newtonian fluid model. In any case the damping distribution matrix $\mathbf{C}_S$ is 'non-proportional' with respect to $\mathbf{M}_S$ and $\mathbf{K}_S$, and more in general it cannot be diagonalized by the modal matrix of the related undamped model ($\mathbf{C}_S = \mathbf{0}$).

This result could be extended without any difficulty to the case of a fluid line coupled with two independent mechanical systems, leading to $H_1 + H_2 + N - 2$ equilibrium equations, or coupled with two different dofs of the same mechanical system, leading to $H + N - 2$ equilibrium equations.

The eigenproblem in Eq. (31) can be rewritten in a state space form:

$$\left[s\begin{bmatrix}\mathbf{C}_S & \mathbf{M}_S\\ \mathbf{M}_S & \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{K}_S & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}_S\end{bmatrix}\right]\begin{bmatrix}\mathbf{y}\\ s\mathbf{y}\end{bmatrix} = \mathbf{0} \quad \Leftrightarrow \quad [s\mathbf{A} + \mathbf{B}]\mathbf{z} = \mathbf{0} \tag{32}$$

In the case of 1-dimensional Newtonian fluid model, Eqs. (32) can be decoupled by solving a conventional eigenproblem. The above described procedure may therefore be conveniently adopted in the case of 1-dimensional Newtonian fluid model, being suitable for easy implementation in the Finite Element Method: the physical dofs connected with a fluid line are 'expanded' to the size of the discretized fluid model, by means of the analytically defined matrices $\mathbf{M}_L$, $\mathbf{C}_L$ and $\mathbf{K}_L$.

In the case of 2-dimensional fluid models, an iterative procedure can be applied to Eq. (32) for computing the eigenvalues. Unfortunately, the simple recursive computation of the Rayleigh's quotient:

$$s_{j+1} \cong -\frac{\mathbf{z}^{\mathrm{T}}\mathbf{B}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{A}_j\mathbf{z}} \tag{33}$$

using the eigenvectors $\mathbf{z}$ of the 1-dimensional model, and introducing at each iteration the actual value s in the eigenvalue-dependent matrix $\mathbf{A}$, in general is not accurate enough (as shown in the following section). Therefore, an appropriate method for the refinement of both the values s and the vectors $\mathbf{z}$ should be adopted. Let s and $\mathbf{z}$ be approximate values, and $s + \Delta s$ and $\mathbf{z} + \Delta\mathbf{z}$ the exact values for them. Then:

$$\begin{cases}[s\mathbf{A} + \mathbf{B}]\mathbf{z} = \boldsymbol{\varepsilon}\\ [(s + \Delta s)\mathbf{A} + \mathbf{B}](\mathbf{z} + \Delta\mathbf{z}) = \mathbf{0}\end{cases} \quad \Longrightarrow \quad \mathbf{A}\mathbf{z}\Delta s + [s\mathbf{A} + \mathbf{B}]\Delta\mathbf{z} \cong -\boldsymbol{\varepsilon} \tag{34}$$

where the elements of second-order smallness are neglected. Without loss of generality, the first component of the vector $\Delta\mathbf{z}$ can be set equal to zero (since $\mathbf{z}$ is determined accurately to within a constant multiplier), yielding a linear system with the same number of equations and unknowns (Δs and $\Delta\mathbf{z}$ without its first component). After solving the system, and introducing the refined value $s + \Delta s$ in the eigenvalue-dependent matrix $\mathbf{A}$, the process can be repeated iteratively (Derwidué's method [16-17]). This is a very simple and generally fast procedure, which works for both Newtonian and non-Newtonian fluid models. The accuracy of the results clearly depends on the accuracy of the discretized model of the fluid line (kind and number of selected shape-functions), paying attention to the conditioning of the coefficient matrices computed in the iterative procedure.

8 Coupling with mechanical systems: solution in the time domain

The direct expression of Frfs through transfer functions, as shown in section 6 Eqs. (24), is a useful tool for selecting an appropriate number of modes for approximating the solution of the coupled fluid-mechanical system. It should be particularly helpful for reducing to a minimum necessary (say $2Z < 2H$) the number of numerically computed eigenvalues s_n:

$$s_n = \sigma_n + \mathrm{i}\Omega_n \tag{35}$$

Approximate expressions for the Irfs can then be written as linear combinations of complex-conjugate exponential functions $\exp(s_n t)$: so the problem reduces to the computation of a set of optimal 'modal' coefficients.

Since analytical expressions of the Frfs are available from Eqs. (24), they can be used for finding a set of optimal 'modal' coefficients by means of a fitting procedure. The imaginary parts Ω_n of the computed eigenvalues s_n are selected as fitting points for comparing the Frfs given by Eqs. (24) and their approximate 'modal' expansions:

$$\sum_{n=1}^{Z} \frac{a_n + \mathrm{i}b_n}{\mathrm{i}\Omega_h - s_n} + \frac{a_n - \mathrm{i}b_n}{\mathrm{i}\Omega_h - s_n^*} = \frac{X_p(\mathrm{i}\Omega_h)}{F_m}, \quad h = 1 \dots 2Z \tag{36}$$

Equations (36) form a linear system of $2Z$ equations in the $2Z$ real unknowns a_n and b_n, yielding a set of $2Z$ approximate 'modal' coefficients:

$$\tilde{\gamma}_n = a_n \pm \mathrm{i}b_n \tag{37}$$

Finally, the analytically aproximated expressions for both the Frfs and Irfs can be written, given the eigenvalues s_n and the 'modal' coefficients in Eq. (37):

$$H(\omega) = \sum_{n=1}^{2Z} \frac{\tilde{\gamma}_n}{\mathrm{i}\omega - s_n}, \quad h(t) = \sum_{n=1}^{2Z} \tilde{\gamma}_n e^{s_n t} \tag{38}$$

These approximations can be acceptable provided the number of selected eigenvalues is sufficient.

9 Numerical application

As represented in Fig. 2, a fluid line of length L closed at one end, at the other end is coupled to a mechanical system consisting of two masses linked by springs and viscous dampers, according to the parameters reported in Tab. 1.

Fluid line	**Mechanical system**
$\beta = 1500$ [Mpa]	$m_1 = m_2 = 0.1$ [Kg]
$\rho = 800$ [Kg m^{-3}]	$c_1 = 0$ [N s m^{-1}], $c_2 = 0.01 \times k_L$ [N s m^{-1}]
$\alpha_2 = 1$	$k_1 = k_2 = 10 \times k_L$ [N m^{-1}]
$\lambda_f = 0$ [s]	$\omega_1 = 424.26081864$ [rad s^{-1}]
$\nu = \mu/\rho = 160$ [cSt]	$\omega_2 = 1110.72924328$ [rad s^{-1}]
$R = 3.5$ [mm]	$s_{1,2} = -67.23056757 \pm \mathrm{i}\, 428.51585446$ [rad s^{-1}]
$L = x_2 - x_1 = 12.25$ [m]	$s_{3,4} = -168.38888144 \pm \mathrm{i}\, 1073.28121743$ [rad s^{-1}]

Table 1: parameters of the coupled systems.

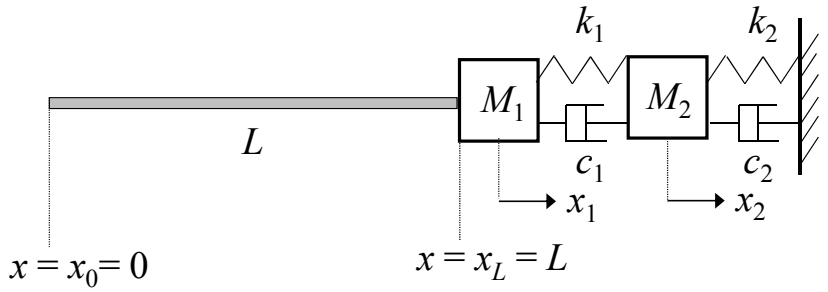

Figure 2: fluid line coupled to a simple mechanical system.

For this system, the following equilibrium equations and boundary conditions can be written in addition to Eqs. (3):

$$\begin{cases} s^2 M_2 X_2 + sc(X_2 - X_1) + k(X_2 - X_1) = 0 \\ s^2 M_1 X_1 + sc(X_1 - X_2) + k(X_1 - X_2) = F_1 = -k_L L \frac{\partial U_a}{\partial x}\Big|_L \\ X_1 = U_a(L) \end{cases} \tag{39}$$

where X_1 and X_2 are the Laplace transforms of the displacements x_1 and x_2.

The natural frequencies and the eigenvalues of modes 1-6, for both the 1- and 2-dimensional fluid Newton models, computed with a zero-finding routine, are reported in Tab. 2.

In Tab. 3 a comparison is presented between the eigenvalues computed by solving the characteristic equation with a zero-finding routine (section 6), and by using the iterative methods described in section 7 for the discretized model (Rayleigh quotient and Derwidué method, with $N = 7$, selected number of dofs for the discretized fluid line). Clearly, the Rayleigh quotient alone (using the eigenvectors of the 1-dimensional fluid model) is not accurate enough. But the results can be significantly improved with a few iterations of the Derwiduè method (using as starting values the results of the Rayleigh quotient iteration).

Mode	ω_n	**1-dimensional** [$b(0)$]	**2-dimensional** [$b(s)$]
1	274.07758475	– 52.77188165 ± i 269.90322106	– 50.76888227 ± i 239.80780285
2	485.64628113	– 61.25776227 ± i 487.37967197	– 66.91557038 ± i 454.32160347
3	751.06290115	– 71.49272997 ± i 756.64878218	– 88.81574156 ± i 701.23196582
4	1023.18111575	– 149.75739471 ± i 1009.31628398	– 147.60446555 ± i 972.38123627
5	1162.07303257	– 68.47401667 ± i 1133.49492441	– 119.51680421 ± i 1085.48015414
6	1449.57750484	– 50.19149507 ± i 1447.76695664	– 97.39091392 ± i 1361.03327703

Table 2: eigenvalues [rad/s], Newton model, and natural frequencies [rad/s]

Mode 1	1-dimensional [Tab. 2]	– 52.77188165 ± i 269.90322106
	Rayleigh 3 iterations	– 51.03295246 ± i 241.45232231
	Derwidué 4 iterations	– 50.76868359 ± i 239.80776990
	Derwidué 8 iterations	– 50.76888227 ± i 239.80780284
	2-dimensional [Tab. 2]	– 50.76888227 ± i 239.80780285
Mode 3	1-dimensional [Tab. 2]	– 71.49272997 ± i 756.64878218
	Rayleigh 3 iterations	– 88.77010416 ± i 701.43042180
	Derwidué 4 iterations	– 88.81574236 ± i 701.23196541
	Derwidué 8 iterations	– 88.81574160 ± i 701.23196594
	2-dimensional [Tab. 2]	– 88.81574156 ± i 701.23196582
Mode 5	1-dimensional [Tab. 2]	– 68.47401667 ± i 1133.49492441
	Rayleigh 3 iterations	– 115.13852882 ± i 1073.64022113
	Derwidué 4 iterations	– 119.51670325 ± i 1085.47984465
	Derwidué 8 iterations	– 119.51678392 ± i 1085.48019834
	2-dimensional [Tab. 2]	– 119.51680421 ± i 1085.48015414

Table 3: Comparison of eigenvalues [rad/s], Newton model.

In the following, Frfs $H(\omega)$ and Irfs $h(t)$ are both computed in $x_{ou} = x_2$, due to forces acting in $x_{in} = x_2$. The Frfs $H(\omega)$ are normalized with respect to the static stiffness:

$$k_{norm} = k_2 + \frac{k_1 k_L}{k_1 + k_L} \tag{40}$$

Figures 3-4 show a comparison between the Frfs $H(\omega)$ computed for the uncoupled mechanical system (as in Tab. 1), the 1- and 2-dimensional Newton models.

Figures 5-6 show the effect of varying the viscosity coefficient μ in the Frf $H(\omega)$ for the 2-dimensional Newton model. The curves are plotted for $0.8\times\mu$, $0.9\times\mu$, $1.0\times\mu$, $1.1\times\mu$, $1.2\times\mu$ respectively, μ as in the reference case of Tab. 1.

Figures 7-8 show a comparison between the Frfs $H(\omega)$ computed for the 2-dimensional Newton and Maxwell models, with $\mu_f = \mu$ in all cases; 1) $\alpha_2 = 1.0$, $\lambda_f = 0$ s; 2) $\alpha_2 = 0.5$, $\lambda_f = 0$ s; 3) $\alpha_2 = 1.0$, $\lambda_f = 0.1\ s^{\alpha_1}$, $\alpha_1 = 0.5$.

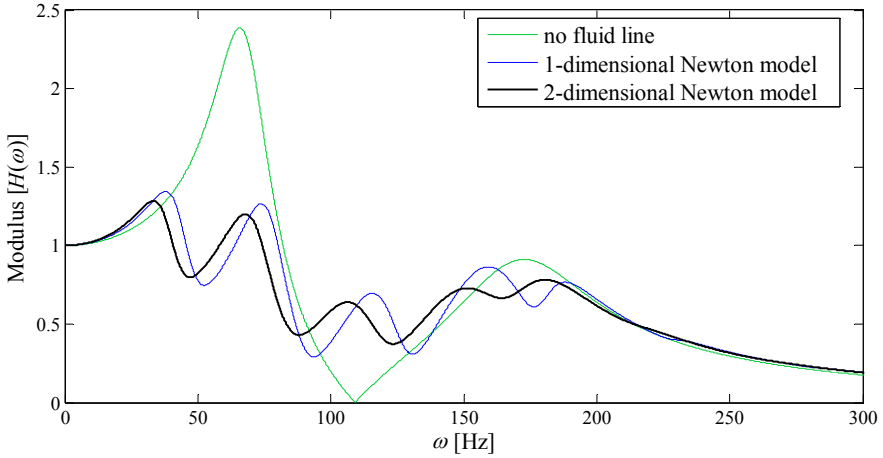

Figure 3: modulus of the Frf $H(\omega)$; uncoupled mechanical system, 1- and 2-dimensional Newton models.

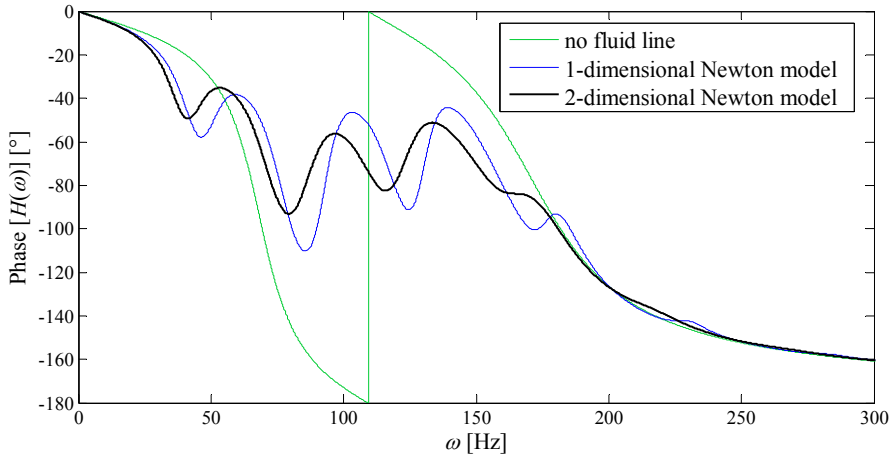

Figure 4: phase of the Frf $H(\omega)$; uncoupled mechanical system, 1- and 2-dimensional Newton models.

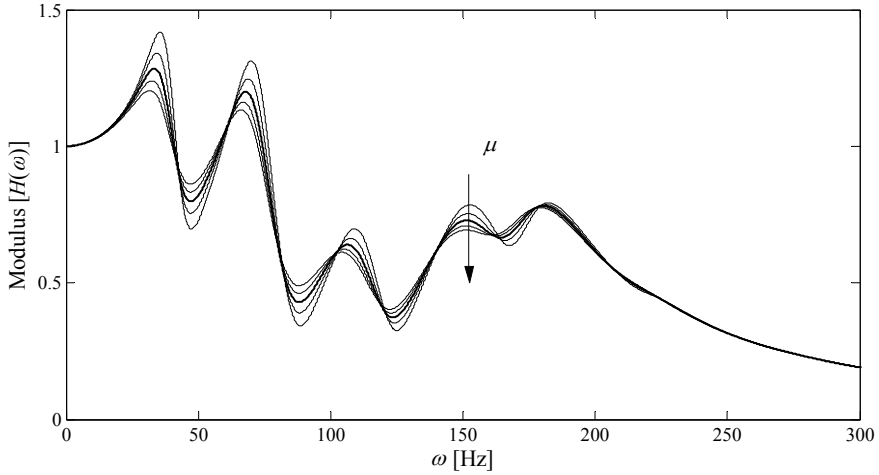

Figure 5: modulus of the Frf $H(\omega)$ for the 2–dimensional Newton model; effect of varying the viscosity μ.

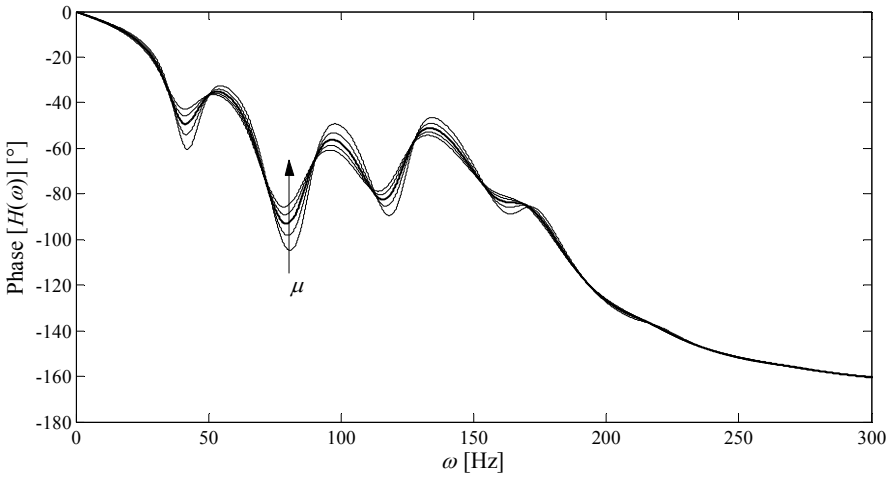

Figure 6: phase of the Frf $H(\omega)$ for the 2-dimensional Newton model; effect of varying the viscosity μ.

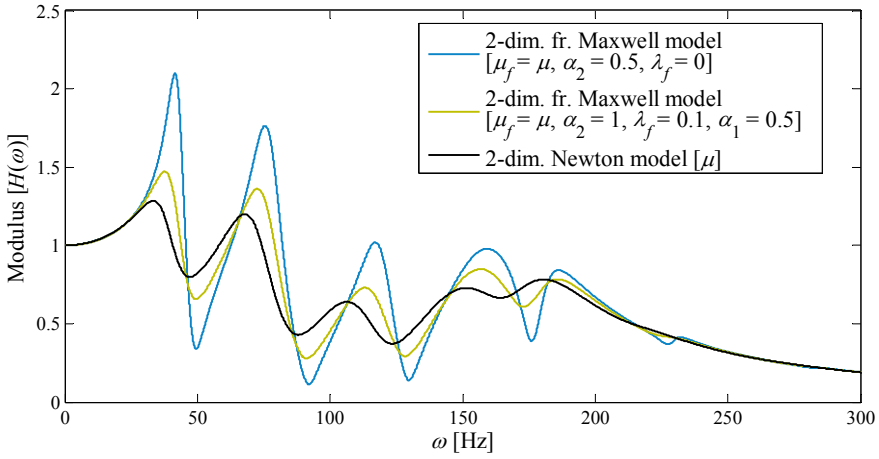

Figure 7: modulus of the Frf $H(\omega)$; comparison between the 2-dimensional Newton and Maxwell models.

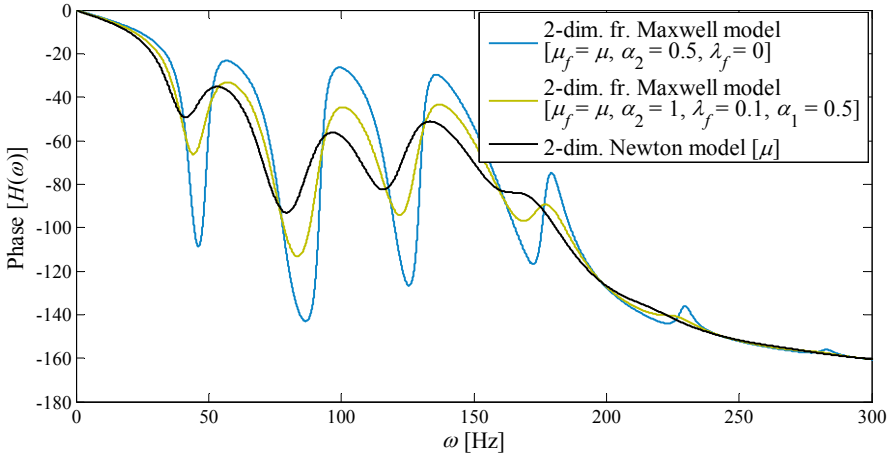

Figure 8: phase of the Frf $H(\omega)$; comparison between the 2-dimensional Newton and Maxwell models.

Figure 9 shows a comparison between the Irfs $h(t)$ computed for the uncoupled mechanical system (as in Tab. 1), and the 1- and 2-dimensional Newton models. In the case of the 1-dimensional Newton model, the differences between the results given by the method of the Inverse Fourier Transform and those given by the discretization method (section 7) are negligible. In the case of the 2-dimensional Newton model, the differences between the results given by the method of the Inverse Fourier Transform and those given by the modal approximation procedure (section 8) can be considered negligible as well: the numerical Irf (computed by means of the inverse fast Fourier transform algorithm, with $\Delta\omega = 2\pi/10$ and $\Delta t = 2.5\times10^{-5}$) is compared with the analytical modal aproximate Irf (condition number of the coefficient matrix $n_{\text{cond}} \approx 22$); the error ε (defined as the absolute value of the difference) between the Irfs is represented in Fig. 10.

Figure 11 shows the effect of varying the viscosity coefficient μ in the Irf $h(t)$ for the 2-dimensional Newton model. The curves are plotted for $0.8\times\mu$, $0.9\times\mu$, $1.0\times\mu$, $1.1\times\mu$, $1.2\times\mu$ respectively, μ as in the reference case of Tab. 1.

Figure 12 shows a comparison between the Irfs $h(t)$ computed for the 2-dimensional Newton and Maxwell models, with $\mu_f = \mu$ in all cases; 1) $\alpha_2 = 1.0$, $\lambda_f = 0$ s; 2) $\alpha_2 = 0.5$, $\lambda_f = 0$ s; 3) $\alpha_2 = 1.0$, $\lambda_f = 0.1\ s^{\alpha_1}$, $\alpha_1 = 0.5$.

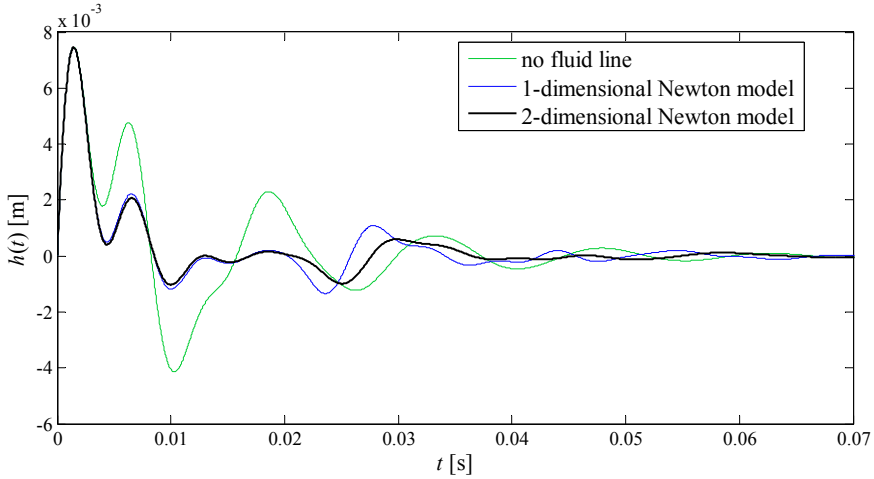

Figure 9: Irf $h(t)$; uncoupled mechanical system, 1- and 2-dimensional Newton models.

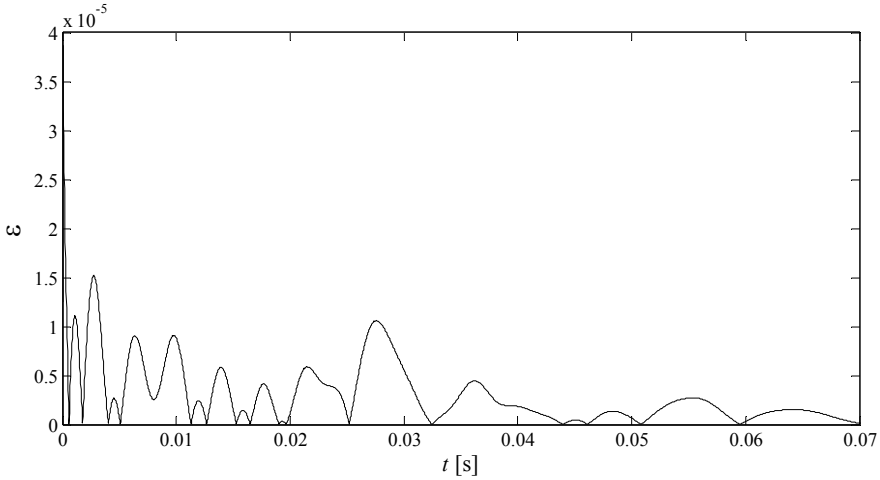

Figure 10: error-difference between numerical and analytical approximate Irf $h(t)$.

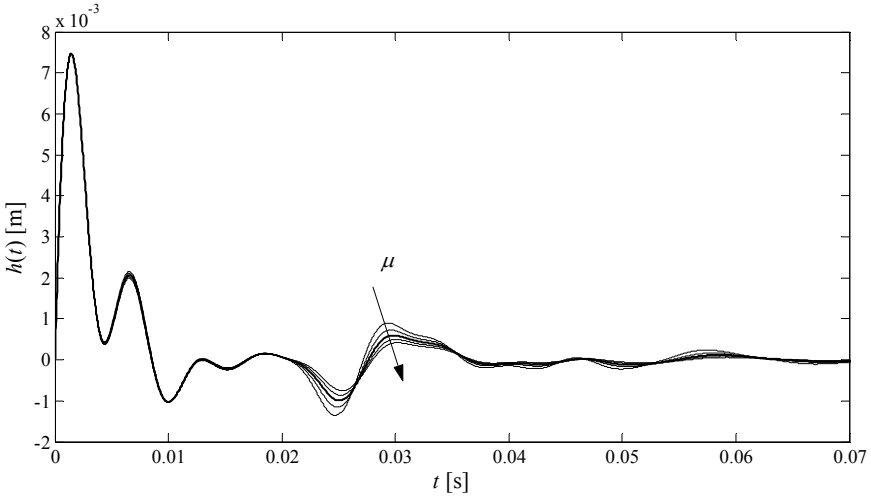

Figure 11: Irf $h(\omega)$ for the 2-dimensional Newton model; effect of varying the viscosity μ.

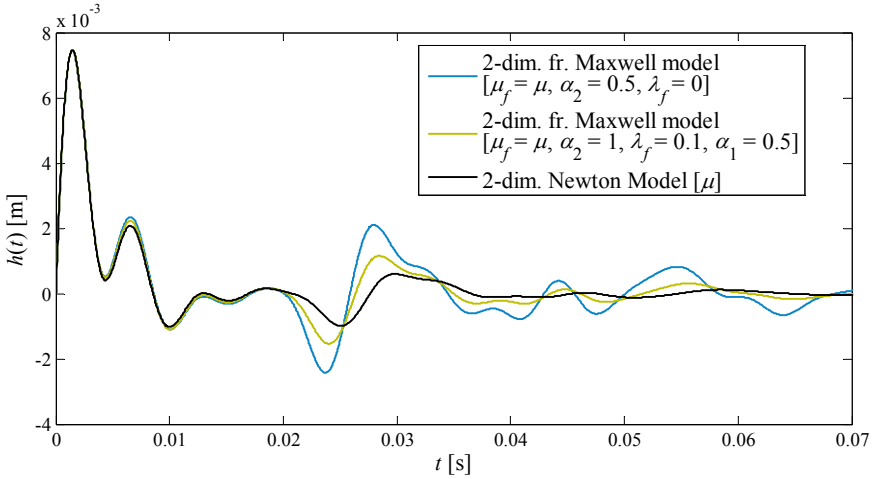

Figure 12: Irf $h(t)$; comparison between the 2-dimensional Newton and Maxwell models.

In any case, strong differences between the 1- and 2-dimensional models can be observed: therefore the 1-dimensional assumption should not be considered for accurate simulations. Note also that the non-integer derivative order α_2 influences both the attenuation and the mode frequencies. A reduction of α_2 reduces the attenuation (primary effect) and increases the mode frequencies (secondary effect). Increasing λ_f or α_1 has the opposite effect of reducing α_2. Clearly, different combinations of the four parameters μ_f, α_2, λ_f and α_1 would allow a fine regulation in controlling the shape of both Frfs and Trfs, which may be of practical interest when fitting experimental data over wide frequency or time intervals.

Conclusions

In this study some procedures were presented for the dynamic analysis of fluid transmission lines with linear viscoelastic fluids coupled to mechanical systems, suitable for fast and accurate simulations of their dynamic responses in both the frequency and time domains.

The transient laminar flow in circular, constant section fluid lines was modeled according to both 1- and 2-dimensional assumptions, and to a non-integer order derivative constitutive law, namely the fractional Maxwell model, of which the Newton model can be considered a particular case.

Two different approaches were adopted, using continuous fluid models (for expressing Frfs in exact analytical form), and discretized fluid models by modal decomposition (for expressing Trfs in approximate analytical form), highlighting the effects of constitutive parametric variations on Frfs and Trfs, which may be of practical interest when fitting experimental data over wide frequency or time intervals. In any case, strong differences between the 1- and 2-dimensional fluid models were observed: therefore the 1-dimensional assumption should not be considered for accurate simulations.

The discretized fluid model is suitable for easy implementation in a FE model of a mechanical system: the computation of eigenvalues and modal coefficients in the case of 2-dimensional linear fluids (Newtonian or non-Newtonian) consist of the solution of a standard linear eigenproblem, followed by the solution of a (generally small) number of linear algebraic systems.

Acknowledgments

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